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The effective action and geometry of closed $N = 2$ strings

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Abstract: $N = 2$ closed strings have been recently divided in [3] to two T-dual families denoted by α and β. In $(2, 2)$ signature both families have one scalar in the spectrum. The scalar in the β-string is known to be a deformation of the target space Kähler potential and the dynamics is that of self-dual gravity. In this paper we compute the effective action of the scalar in the α-string. The scalar is a deformation of a potential that determines the metric, torsion and dilaton. The scalar is free and the dynamics is that of a self-dual curvature with torsion. The result is in agreement with a σ-model computation of Hull.

Keywords: Superstrings and Heterotic Strings, Extended Supersymmetry.

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1. Introduction

Closed $N = 2$ strings [1] possess local $\mathcal{N} = 2$ supersymmetry on the string worldsheet. Critical $N = 2$ strings have a four-dimensional target space. The supersymmetric structure implies that the target space has a complex structure. Therefore it must be of signature $(4, 0)$ or $(2, 2)$. In $(4, 0)$ signature there are no propagating degrees of freedom in the $N = 2$ string spectrum. In $(2, 2)$ signature there is only one massless scalar in the spectrum and the infinite tower of massive excitations of the string is absent. The effective action of this scalar has been computed in [2] which suggested its interpretation as a deformation of the target space Kähler potential. It was argued in [2] that the $N = 2$ strings may describe a quantum theory of self-dual Einstein gravity in four dimensions.

In [3] the $N = 2$ closed strings have been divided into two families denoted by α and β. Consider $N = 2$ strings in a flat background. In order to construct the $N = 2$ string we need to gauge the $N = 2$ superconformal algebra (SCA) on the worldsheet. More precisely we have two copies of the $N = 2$ algebra to consider: the left and right sectors. The free field representation of the (left) $N = 2$ SCA takes the form

$$\begin{aligned} T &= -\frac{1}{2}\eta_{IJ}\partial x^I \partial x^J - \frac{1}{4}\eta_{IJ}\left(\partial\psi^I\psi^J + \partial\psi^J\psi^I\right) , \\ J &= \frac{i}{2}\mathcal{J}^L_{IJ}\psi^I\psi^J , \\ G^{\pm} &= \frac{i}{2}\left(\eta_{IJ} \pm i\mathcal{J}^L_{IJ}\right)\psi^I\partial x^J . \end{aligned} \tag{1.1}$$

Here $I, J = 1, \ldots, 4$ denote the indices of the target space in a real basis. The metric is given by $\eta_{IJ} = \mathrm{diag}(-1, -1, +1, +1)$. $\mathcal{J}^L_{IJ}$ is a Kähler form related to the complex structure

$\mathcal{J}^K_J$ by $\mathcal{J}_{IJ} = \eta_{IK}\mathcal{J}^K_J$, and the index L refers to the left sector.[1] Similarly, we have the SCA generators in the right sector with a complex structure $\mathcal{J}^R$. The (conventional) $N = 2$ string denoted by β-string in [3] is defined by having the same complex structure in the left and right sectors $\mathcal{J}^L = \mathcal{J}^R$. On the other hand, $N = 2$ string denoted by α-string in [3] has different complex structures in the left and right sectors.[2] In fact the β- and α-strings define two families of $N = 2$ strings related by T-duality [3].

In $(2, 2)$ signature both families have one scalar in the spectrum. The scalar in the β-string is, as noted above, a deformation of the target space Kähler potential and the dynamics is that of self-dual gravity. The aim of this paper is to compute the exact effective action of the scalar in the α-string. The scalar is a deformation of a potential that determines the target space metric, torsion and dilaton [5]. The dynamics is that of a self-dual curvature with torsion [6].

The paper is organized as follows. In section 2, we consider the worldsheet description of $N = 2$ strings. We present the σ-model lagrangian description of the α-string using a chiral and a twisted chiral superfields and construct the vertex operators. In section 3, we compute the genus zero three-point and four-point scattering amplitudes of the $N = 2$ strings scalar. In section 4 we show, based on sections 2 and 3, that the α-string scalar is free and that the dynamics is that of a self-dual curvature with torsion. This has been anticipated in [6] based on σ-model and conformal anomaly analysis. In section 5 we construct an example of a gravitational α-string background, based on the space transverse to NS5-branes.

2. Worldsheet description of $N = 2$ strings

In this section we will discuss in some detail the worldsheet description of the β- and α-strings. We consider the σ-model lagrangian and the vertex operators.

2.1 Complex structure and $N = 2$ SCA

In the following we will review some aspects of the complex structures on $\mathbf{R}^{2,2}$ that are relevant to the generators of $N = 2$ SCA. In the real basis $x^I = (x^1, x^2, x^3, x^4)$, the metric is given by $\eta_{IJ} = \mathrm{diag}(-1, -1, +1, +1)$. We define a complex structure

$$\mathcal{J}^I{}_J = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} . \tag{2.1}$$

In the complex basis

$$z^1 = \frac{x^1 + ix^2}{\sqrt{2}} , \qquad z^2 = \frac{x^3 + ix^4}{\sqrt{2}} , \tag{2.2}$$

the metric reads $\eta_{i\bar{j}} = \mathrm{diag}(-1, +1)$, $i, \bar{j} = 1, 2$. In this basis, the complex structure $\mathcal{J}^I{}_J$ is diagonal:

$$\begin{aligned} \mathcal{J}(z^1) &= -iz^1 , \qquad & \mathcal{J}(\bar{z}^{\bar{1}}) &= +i\bar{z}^{\bar{1}} , \\ \mathcal{J}(z^2) &= -iz^2 , \qquad & \mathcal{J}(\bar{z}^{\bar{2}}) &= +i\bar{z}^{\bar{2}} . \end{aligned} \tag{2.3}$$

[1] For more details see next section and [3].

[2] For an earlier discussion see [4].

The Kähler form $\mathcal{J}_{IJ} = \eta_{IK}\mathcal{J}^K_J$ is given in the real basis by

$$\mathcal{J}_{IJ} = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}. \tag{2.4}$$

For later reference, we define the quadratic form in momenta

$$k^I_i \mathcal{J}_{IJ} k^J_j = ic_{ij}\,, \tag{2.5}$$

where in the complex basis

$$c_{ij} = k_i \cdot \bar{k}_j - \bar{k}_i \cdot k_j\,, \tag{2.6}$$

and $k_i \cdot \bar{k}_j \equiv \eta_{m\bar{n}} k^m_i \bar{k}^{\bar{n}}_j$. For the on-shell momenta $k_i,\ i = 1,2,3,4$ with $k^2_i = 0,\ k_1 + k_2 + k_3 + k_4 = 0$, c_{ij} obey the identities [2]

$$\frac{c_{12}c_{34}}{s} + \frac{c_{23}c_{41}}{t} = u\,, \qquad \frac{c_{21}c_{34}}{s} + \frac{c_{13}c_{42}}{u} = t\,, \qquad \frac{c_{13}c_{24}}{u} + \frac{c_{32}c_{41}}{t} = s\,, \tag{2.7}$$

where $s = -k_1 \cdot k_2 \equiv -(k_1 \cdot \bar{k}_2 + \bar{k}_1 \cdot k_2)$, $t = -k_2 \cdot k_3$, $u = -k_1 \cdot k_3$.

We will also need a second complex (and Kähler) structure, which in the real basis take the form

$$\widetilde{\mathcal{J}}^I{}_J = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \qquad \widetilde{\mathcal{J}}_{IJ} = \eta_{IK}\widetilde{\mathcal{J}}^K_J = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}. \tag{2.8}$$

In the complex basis, the complex structure is given by

$$\begin{aligned} \widetilde{\mathcal{J}}(z^1) &= -iz^1\,, \qquad & \widetilde{\mathcal{J}}(\bar{z}^{\bar{1}}) &= +i\bar{z}^{\bar{1}}\,, \\ \widetilde{\mathcal{J}}(z^2) &= +iz^2\,, \qquad & \widetilde{\mathcal{J}}(\bar{z}^{\bar{2}}) &= -i\bar{z}^{\bar{2}}\,. \end{aligned} \tag{2.9}$$

The Kähler form reads

$$\widetilde{\mathcal{J}}_{i\bar{j}} = -i\mathbf{1}\,, \qquad \widetilde{\mathcal{J}}_{\bar{i}j} = +i\mathbf{1}\,. \tag{2.10}$$

We define also

$$\widetilde{c}_{ij} = -ik^I_i \widetilde{\mathcal{J}}_{IJ} k^J_j = -k^1_i \bar{k}^1_j - k^2_i \bar{k}^2_j + \bar{k}^1_i k^1_j + \bar{k}^2_i k^2_j\,, \tag{2.11}$$

which obeys an equation similar to (2.7), for four on-shell momenta.

The β-string in $\mathbf{R}^{2,2}$ is defined by the choice[3]

$$\mathcal{J}^L_{IJ} = \mathcal{J}^R_{IJ} = \mathcal{J}_{IJ}\,, \tag{2.12}$$

in (1.1), while for the α-string

$$\mathcal{J}^L_{IJ} = \mathcal{J}_{IJ}\,, \qquad \mathcal{J}^R_{IJ} = \widetilde{\mathcal{J}}_{IJ}\,. \tag{2.13}$$

The two $N = 2$ strings are related by a T-duality along a spatial direction.

[3]More precisely, there is a parameter space of β-strings corresponding to the choice of the complex structures in the $N = 2$ SCA, see [3]. The results that will be presented later remain valid for any choice of the complex structure $\mathcal{J}_L = \mathcal{J}_R$.

2.2 A σ-model lagrangian description

Here we will consider the two-dimensional σ-model lagrangian description of the $N = 2$ strings. We begin by reviewing the $\mathcal{N} = (2,2)$ superfield formulation in euclidean two-dimensional space-time with the complex coordinates z and $\bar{z}$ (see e.g. [5]). The complex supercovariant derivatives for the left and right movers $D_\pm$ and $\bar{D}_\pm$ are defined by

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\mp\partial_z\,, \qquad \bar{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + \bar{\theta}^\mp\partial_{\bar{z}}\,, \tag{2.14}$$

where $\theta^\pm$ are the complex fermionic coordinates in the superspace. Denote $z^\pm, \bar{z}^\pm$ by

$$z^\pm = z \pm \theta^2\,, \qquad \bar{z}^\pm = \bar{z} \pm \bar{\theta}^2\,, \tag{2.15}$$

with $\theta^2 = \theta^+\theta^-$, $\bar{\theta}^2 = \bar{\theta}^-\bar{\theta}^+$. It is easy to see that

$$D_\pm z^\mp = 0\,. \tag{2.16}$$

A chiral superfield X is defined by

$$D_+X = \bar{D}_+X = 0\,, \tag{2.17}$$

and an anti-chiral superfield $\bar{X}$ by

$$D_-\bar{X} = \bar{D}_-\bar{X} = 0\,. \tag{2.18}$$

In components we have

$$\begin{aligned} X &= X(z^-,\bar{z}^-,\theta^-,\bar{\theta}^-) \\ &= x(z^-,\bar{z}^-) + \sqrt{2}\,\theta^-\psi_L(z^-,\bar{z}^-) + \sqrt{2}\,\bar{\theta}^-\psi_R(z^-,\bar{z}^-) + 2\theta^-\bar{\theta}^-F(z^-,\bar{z}^-)\,, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \bar{X} &= \bar{X}(z^+,\bar{z}^+,\theta^+,\bar{\theta}^+) \\ &= \bar{x}(z^+,\bar{z}^+) + \sqrt{2}\,\theta^+\bar{\psi}_L(z^+,\bar{z}^+) - \sqrt{2}\,\bar{\theta}^+\bar{\psi}_R(z^+,\bar{z}^+) + 2\theta^+\bar{\theta}^+\bar{F}(z^+,\bar{z}^+)\,. \end{aligned} \tag{2.20}$$

Since $\{D_\pm, \bar{D}_\pm\} = 0$, one can define in two dimensions twisted chiral superfields $\widetilde{X}$ by

$$D_+\widetilde{X} = \bar{D}_-\widetilde{X} = 0\,, \tag{2.21}$$

and an anti-twisted chiral $\bar{\widetilde{X}}$ superfield by

$$D_-\bar{\widetilde{X}} = \bar{D}_+\bar{\widetilde{X}} = 0\,. \tag{2.22}$$

In components we have

$$\begin{aligned} \widetilde{X} &= \widetilde{X}(z^-,\bar{z}^+,\theta^-,\bar{\theta}^+) \\ &= x(z^-,\bar{z}^+) + \sqrt{2}\,\theta^-\psi_L(z^-,\bar{z}^+) + \sqrt{2}\,\bar{\theta}^+\psi_R(z^-,\bar{z}^+) + 2\theta^-\bar{\theta}^+F(z^-,\bar{z}^+)\,, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\bar{\widetilde{X}} &= \bar{\widetilde{X}}(z^+, \bar{z}^-, \theta^+, \bar{\theta}^-) \\ &= \bar{x}(z^+, \bar{z}^-) + \sqrt{2}\,\theta^+ \bar{\psi}_L(z^+, \bar{z}^-) - \sqrt{2}\,\bar{\theta}^- \bar{\psi}_R(z^+, \bar{z}^-) + 2\theta^+\bar{\theta}^- F(z^+, \bar{z}^-)\,. \end{aligned} \tag{2.24}$$

T-duality along $X + \bar{X}$ exchanges chiral (anti-chiral) superfields and twisted (anti-twisted) chiral superfields. In the complex structure language, the left movers complex structure does not change but the right movers complex structure does ($\mathcal{J} \leftrightarrow \widetilde{\mathcal{J}}$).

The $N = 2$ σ-model action for the β-string in a flat background is

$$S^0_\beta = \int d^4x d^4\theta K_0 \left(X^1, \bar{X}^1, X^2, \bar{X}^2\right), \tag{2.25}$$

where $X^i, i = 1, 2$ are chiral superfields and K_0 is the Kähler potential for flat $(2, 2)$ space

$$K_0(X^1, \bar{X}^1, X^2, \bar{X}^2) = -X^1\bar{X}^1 + X^2\bar{X}^2 \sim -X^1\bar{X}^1 + \frac{1}{2}(X^2 + \bar{X}^2)^2\,. \tag{2.26}$$

In the last equivalence we used the freedom of adding holomorphic and antiholomorphic terms to the Kähler potential without changing the metric $g_{i\bar{j}} = \partial_{X^i}\partial_{\bar{X}^j} K_0$.

The $N = 2$ σ-model action for the α-string in a flat background is

$$S^0_\alpha = \int d^4x d^4\theta \widehat{K}_0(X^1, \bar{X}^1, \widetilde{X}^2, \bar{\widetilde{X}}{}^2)\,, \tag{2.27}$$

where X^1 is a chiral superfield and $\widetilde{X}^2$ is a twisted chiral superfield. $\widehat{K}_0$ is the Legendre transform of K_0 with respect to $X^2 + \bar{X}^2$ [3] (see [7, 8, 9]). Thus,

$$\widetilde{X}^2 + \bar{\widetilde{X}}{}^2 = \frac{\partial K_0}{\partial(X^2 + \bar{X}^2)} = X^2 + \bar{X}^2\,, \tag{2.28}$$

and

$$\widehat{K}_0\left(X^1, \bar{X}^1, \widetilde{X}^2, \bar{\widetilde{X}}{}^2\right) = K_0 - (X^2 + \bar{X}^2)\left(\widetilde{X}^2 + \bar{\widetilde{X}}{}^2\right) = -X^1\bar{X}^1 - \frac{1}{2}\left(\widetilde{X}^2 + \bar{\widetilde{X}}{}^2\right)^2. \tag{2.29}$$

The target space metric, torsion and dilaton are encoded in the function $\widehat{K}_0$ [5]

$$\hat{g}_{1\bar{1}} = \frac{\partial^2 \hat{K}_0}{\partial X^1 \partial \bar{X}^1} = -1\,, \qquad \hat{g}_{2\bar{2}} = -\frac{\partial^2 \hat{K}_0}{\partial \widetilde{X}^2 \partial \bar{\widetilde{X}}{}^2} = +1\,. \tag{2.30}$$

In flat space the torsion and dilaton vanish and we have

$$B_{1\bar{2}} = \frac{\partial^2 \hat{K}_0}{\partial X^1 \partial \bar{\widetilde{X}}{}^2} = 0\,, \qquad B_{2\bar{1}} = \frac{\partial^2 \hat{K}_0}{\partial \widetilde{X}^2 \partial \bar{X}^1} = 0\,, \tag{2.31}$$

and

$$\varphi = \frac{1}{2} \log \hat{g}_{2\bar{2}} = 0\,. \tag{2.32}$$

Note that in order to get the dilaton equation (2.32), the dilaton term [10] should be added to the σ-model action S^0_α.

2.3 Vertex operators

The vertex operators for the massless scalar in the β- and α-strings are given by

$$V_\beta(k) = e^{i(\bar{k}^1 X^1 + \bar{k}^2 X^2 + k^1 \bar{X}^1 + k^2 \bar{X}^2)}\,, \tag{2.33}$$

and

$$V_\alpha(k) = e^{i(\bar{k}^1 X^1 + \bar{k}^2 \widetilde{X}^2 + k^1 \bar{X}^1 + k^2 \bar{\widetilde{X^2}})}\,, \tag{2.34}$$

respectively. The ghost part will be treated separately. The vertex operator in the $(-1,-1)$-picture comes from the lowest component of the superfields and is the same for the β- and α-strings

$$V_L^{(-1,-1)}(z) = e^{ik\cdot x_L(z)}\,, \qquad V_R^{(-1,-1)}(\bar{z}) = e^{ik\cdot x_R(\bar{z})}\,, \tag{2.35}$$

with $k^2 = 0$. Note that we use x as the lowest component of the superfield X.

In order to obtain the vertex operators of higher superconformal ghost numbers we need to use the picture-changing operations [11] (see also [12]), that are implemented by acting on the vertex operators with the worldsheet supercharges given in (1.1). Using the OPE's of the free fields of the form

$$x_L^I(z) x_L^J(w) \sim -\eta^{IJ} \log(z-w)\,, \qquad \psi_L^I(z)\psi_L^J(w) \sim -\frac{\eta^{IJ}}{z-w}\,, \tag{2.36}$$

it follows that

$$G_L^{\pm}(z) V_L^{(-1,-1)}(0) \sim \frac{1}{z}\left(V_L^{(0,-1)}(0),\, V_L^{(-1,0)}(0)\right), \tag{2.37}$$

from which

$$V_L^{(0,-1)}(z) = \frac{1}{2}(kJ_-^L \psi_L(z))\, e^{ik\cdot x_L(z)}\,, \qquad V_L^{(-1,0)}(z) = \frac{1}{2}(kJ_+^L \psi_L(z))\, e^{ik\cdot x_L(z)}\,. \tag{2.38}$$

Here $J_\pm^L = \eta \pm i\mathcal{J}^L$. Also

$$G_R^{\pm}(\bar{z}) V_R^{(-1,-1)}(0) \sim \frac{1}{\bar{z}}\left(V_R^{(0,-1)}(0),\, V_R^{(-1,0)}(0)\right), \tag{2.39}$$

from which

$$V_R^{(0,-1)}(\bar{z}) = \frac{1}{2}(kJ_-^R \psi_R(\bar{z}))\, e^{ik\cdot x_R(\bar{z})}\,, \qquad V_R^{(-1,0)}(\bar{z}) = \frac{1}{2}(kJ_+^R \psi_R(\bar{z}))\, e^{ik\cdot x_R(\bar{z})}\,. \tag{2.40}$$

Here we define $J_\pm^R = \eta \pm i\mathcal{J}^R$. Finally, the vertex operator of $(0,0)$-picture is

$$G_L^-(z) V_L^{(0,-1)}(0) - G_L^+(z) V_L^{(-1,0)}(0) \sim \frac{1}{z}\, V_L^{(0,0)}(0)\,, \tag{2.41}$$

from which

$$V_L^{(0,0)}(z) = \left(-k\mathcal{J}^L \partial x_L + \frac{1}{2}\left(kJ_+^L \psi_L\right)\left(kJ_-^L \psi_L\right)\right) e^{ik\cdot x_L}\,. \tag{2.42}$$

The right sector vertex operator of the same picture takes the same form with $L \to R$.

As discussed in [2], the scalar ϕ of the β-string is a deformation of the Kähler structure (2.26) as $K_0 \to K_0 + \phi$. What is the interpretation of the scalar ϕ of the α-string? A generating function for the scattering amplitudes is

$$\left\langle \exp\left(\left(\int d^2z d^4\theta\, \phi(X^1, \bar{X}^1, \widetilde{X}^2, \widetilde{\bar{X}^2})\right)\right)\right\rangle , \tag{2.43}$$

where

$$\phi\left(X^1, \bar{X}^1, \widetilde{X}^2, \widetilde{\bar{X}^2}\right) = \int d^4k\, \widetilde{\phi}(k) V_\alpha(k) , \tag{2.44}$$

and $V_\alpha(k)$ is the scalar vertex operator (2.34). Thus, ϕ is a deformation of $\hat{K}_0$ in (2.30) $\hat{K}_0 \to \hat{K}_0 + \phi$.

In the next section, we will compute the scattering amplitudes of α-string and derive the effective action for ϕ.

3. Scattering amplitudes

In this section, we compute the three and four point amplitudes of $N = 2$ strings on a sphere. Let us begin with the three point amplitude. Recall that the total superconformal ghost number of the amplitude on the sphere is two. The amplitude takes the form

$$\mathcal{A}_3 = \mathcal{A}_3^L \, \mathcal{A}_3^R , \tag{3.1}$$

where $\mathcal{A}_3^L$ ($\mathcal{A}_3^R$) is the left (right) sector contribution

$$\begin{aligned} \mathcal{A}_3^L &= \langle V_L^{(-1,-1)}(z_1)\, V_L^{(-1,-1)}(z_2)\, V_L^{(0,0)}(z_3)\rangle \times \\ &\times \langle c_L(z_1)\, c_L(z_2)\, c_L(z_3)\rangle \, \langle e^{-\varphi_L^+(z_1)}\, e^{-\varphi_L^+(z_2)}\rangle \, \langle e^{-\varphi_L^-(z_1)}\, e^{-\varphi_L^-(z_2)}\rangle . \end{aligned} \tag{3.2}$$

Here c_L is the spin -1 conformal ghost, and $\varphi^\pm$ come from the bosonization of the two sets of superconformal ghosts $\beta^\pm, \gamma^\pm$. To evaluate the ghost part in $\mathcal{A}_3$, we use the formulas

$$\begin{aligned} \langle c_L(z_1) c_L(z_2) c_L(z_3)\rangle &= (z_1 - z_2)(z_2 - z_3)(z_1 - z_3) , \\ \left\langle e^{-\varphi_L^\pm(z_1)} e^{-\varphi_L^\pm(z_2)} \right\rangle &= \frac{1}{z_1 - z_2} . \end{aligned} \tag{3.3}$$

By setting, say, $z_1 = \infty,\ z_2 = 0,\ z_3 = 1$, we obtain

$$\mathcal{A}_3 = c_{ij}^L \, c_{ij}^R , \tag{3.4}$$

where

$$c_{ij}^{L(R)} = -ik_i^I \mathcal{J}_{IJ}^{L(R)} k_j^J , \tag{3.5}$$

and $i \neq j$. For β-string, $\mathcal{J}^L = \mathcal{J}^R = \mathcal{J}$ so that

$$\mathcal{A}_3^\beta = (c_{ij})^2 , \tag{3.6}$$

where

$$c_{ij} = -ik_i^I \mathcal{J}_{IJ} k_j^J , \tag{3.7}$$

as in (2.5) and (2.6). This reproduces the result of [2].

For α-string, $\mathcal{J}^L = \mathcal{J}, \mathcal{J}^R = \widetilde{\mathcal{J}}$ and

$$\mathcal{A}_3^\alpha = c_{ij}\, \widetilde{c}_{ij} \,, \tag{3.8}$$

where

$$\widetilde{c}_{ij} = -ik_i^I \widetilde{\mathcal{J}}_{IJ} k_j^J \,, \tag{3.9}$$

as in (2.11).

The two amplitudes $\mathcal{A}_3^\beta$ and $\mathcal{A}_3^\alpha$ are related by T-duality along one of two spatial directions.[4] Consider, for instance, a T-duality along the x^4 direction, which is implemented by $k_R^4 \to -k_R^4$. Here k_R is the momentum of a string in the right sector. It is easy to see that

$$c_{ij}(k^1, k^2, k^3, -k^4) = \widetilde{c}_{ij}(k^1, k^2, k^3, +k^4) \,. \tag{3.10}$$

The amplitude (3.6) is reproduced by an effective action for the scalar [2] satisfying the Plebanski equation [13]. We turn to the amplitude (3.8) and ask what is the effective action for the scalar that reproduces it. Here we find a somewhat surprising result.

Consider three on-shell momenta k_i obeying $k_1 + k_2 + k_3 = 0$. It is useful to work in the complex basis of section 2, where the on-shell momenta take the form $k = (k^l, \bar{k}^{\bar{l}})$ with $k_i^1 = k_i e^{i\theta_i}$, $k_i^2 = k_i e^{i\phi_i}$ and k_i, θ_i, ϕ_i are real. Momentum conservation implies the relation

$$\cos(\theta_i - \theta_j) = \cos(\phi_i - \phi_j) \,. \tag{3.11}$$

We can recast the amplitude as

$$c_{ij}\widetilde{c}_{ij} = 4(k_i k_j)^2 \left[\cos^2(\theta_i - \theta_j) - \cos^2(\phi_i - \phi_j)\right] , \tag{3.12}$$

and hence $\mathcal{A}_3^\alpha = 0$. This means that the effective action of the scalar of the α-string has no three-point interaction.

Let us turn now to the computation of the four-point amplitude. Consider

$$\mathcal{A}_4 = \int d^2z \mathcal{A}_4^L\, \mathcal{A}_4^R \,, \tag{3.13}$$

where $\mathcal{A}_4^L$ comes from the left sector

$$\begin{aligned} \mathcal{A}_4^L = & \left\langle V_L^{(-1,-1)}(z_1)\, V_L^{(-1,-1)}(z_2)\, V_L^{(0,0)}(z_3)\, V_L^{(0,0)}(z) \right\rangle \times \\ & \times \langle c_L(z_1)\, c_L(z_2)\, c_L(z_3) \rangle \left\langle e^{-\varphi_L^+(z_1)}\, e^{-\varphi_L^+(z_2)} \right\rangle \left\langle e^{-\varphi_L^-(z_1)}\, e^{-\varphi_L^-(z_2)} \right\rangle , \end{aligned} \tag{3.14}$$

and a similar expression for the right sector. To evaluate the amplitude, we need the following formulas that involve the complex structures

$$J_\pm \eta^{-1} J_\pm = 2J_\pm \,, \qquad J_\pm \eta^{-1} J_\mp = 0 \,, \tag{3.15}$$

[4] Under T-duality along a time-like direction the amplitudes are equal up to a sign.

where $J_\pm = \eta \pm i\mathcal{J}$, and the integral formula

$$\int d^2z\, z^{\alpha+n_1}\bar{z}^{\alpha+n_2}(1-z)^{\beta+m_1}(1-\bar{z})^{\beta+m_2} =$$
$$= \frac{\sin(\pi\alpha)\sin(\pi\beta)}{\sin(\pi(\alpha+\beta))}\frac{\Gamma(\alpha+n_1+1)\Gamma(\alpha+n_2+1)\Gamma(\beta+m_1+1)\Gamma(\beta+m_2+1)}{\Gamma(\alpha+\beta+n_1+m_1+2)\Gamma(\alpha+\beta+n_2+m_2+2)}\,. \quad (3.16)$$

Setting, say, $z_1 = 1$, $z_2 = \infty$, $z_3 = 0$, it is straightforward to show that

$$\mathcal{A}_4 = \pi\, F^L F^R \frac{\Gamma(1-s)\,\Gamma(1-t)\,\Gamma(1-u)}{\Gamma(s)\,\Gamma(t)\,\Gamma(u)}\,, \quad (3.17)$$

with

$$F^{L(R)} = 1 - \frac{c_{12}^{L(R)}c_{34}^{L(R)}}{su} - \frac{c_{23}^{L(R)}c_{41}^{L(R)}}{tu}\,. \quad (3.18)$$

Here $s = -k_1\cdot k_2$, $t = -k_2\cdot k_3$, $u = -k_1\cdot k_3$. Using the (2.7) we find that

$$\mathcal{A}_4^\alpha = \mathcal{A}_4^\beta = 0\,. \quad (3.19)$$

Consider now the higher-point amplitudes. As discussed in [2], these amplitudes, if nonzero, will have infinite number of poles corresponding to unphysical massive string states, and therefore should vanish. Alternatively, it has been argued in [14, 15] that for $N = 2$ strings in a flat background, all the scattering amplitudes vanish except the sphere three-point amplitude. This stronger claim implies that the α-string scalar is free to all orders in string perturbation theory. Note that, as argued in [16], the inclusion of contributions from sectors of nonzero worldsheet U(1) instanton numbers does not modify the result.

4. Effective action and geometry

In this section we discuss the effective action and geometry of the $N = 2$ strings. As shown in [2], the effective action of the β-string scalar is given by

$$S_\beta = \int \left(\partial\phi\bar{\partial}\phi + \frac{1}{3}\,\phi\,\partial\bar{\partial}\phi\wedge\partial\bar{\partial}\phi\right). \quad (4.1)$$

This effective action reproduces the correct three-point and four-point amplitudes of the string computation. It has not been verified yet that all the higher-order amplitudes arising from (4.1) vanish.[5]

Recall that ϕ is a deformation of the flat space Kähler potential. The field equation for ϕ is the Ricci flatness condition. Thus, the background is Kähler and Ricci-flat, which in four dimensions is equivalent to the curvature being self-dual.

Consider next the α-string scalar. We have seen that all the $n(\geq 3)$-point amplitudes vanish. This implies that α-string scalar is free and the effective action is

$$S_\alpha = \int \partial\phi\bar{\partial}\phi\,. \quad (4.2)$$

[5]For partial results in the open and closed strings cases see [17].

We have seen that ϕ is a deformation of the potential $\widehat{K}_0$ in (2.27), so that the σ-model action is

$$S = \int d^2\sigma d^4\theta\, \widehat{K}(X, \bar{X}, \widetilde{X}, \bar{\widetilde{X}})\,, \tag{4.3}$$

where $\widehat{K} = \widehat{K}_0 + \phi$. The geometry of the theories described by (4.3) has been studied in [5]. Define

$$\begin{aligned} g_{X\bar{X}} &= \partial_X \partial_{\bar{X}} \widehat{K}\,, & g_{\widetilde{X}\bar{\widetilde{X}}} &= -\partial_{\widetilde{X}} \partial_{\bar{\widetilde{X}}} \widehat{K}\,, \\ B_{X\bar{\widetilde{X}}} &= \partial_X \partial_{\bar{\widetilde{X}}} \widehat{K}\,, & B_{\widetilde{X}\bar{X}} &= \partial_{\widetilde{X}} \partial_{\bar{X}} \widehat{K}\,, \end{aligned} \tag{4.4}$$

then by examining the scalar component of the action one finds that g_{IJ} is the metric of the target space and B_{IJ} is a two-form field.

One defines a connection with torsion by

$$\Gamma^{\pm K}{}_{IJ} = \Gamma^K{}_{IJ} \mp H^K_{IJ}\,, \tag{4.5}$$

where $H = dB$. Conformal invariance requires the self-duality condition of the Riemann tensor with torsion (4.5) $R^{\pm}_{IJKL}$ [6]

$$R^{\pm}_{IJKL} = \frac{1}{2}\epsilon_{KLMN} R^{\pm}_{IJ}{}^{MN}\,, \tag{4.6}$$

and

$$R^{+}_{IJKL} = R^{-}_{KLIJ}\,. \tag{4.7}$$

The derivation of the result that $\widehat{K}$ is a free scalar using the worldsheet computations that we performed is in agreement with [6] that predicted this based on the conformal invariance of the σ-model. We note that while in the σ-model the result is expected to hold to all orders in α', here we expect that it holds to all orders in the string loop expansion.

5. An α-string background

As an example we show that the four-dimensional transverse part of an NS5-brane background [18] solves the field equations of the α-string. This is expected since the NS5-branes have been argued to be T-dual to ALE spaces [19], which are solutions of the β-string field equations. Thus, by T-duality we expect the NS5-branes to solve the α-string field equations.

We consider first the signature $(4,0)$. The transverse part to the NS5-branes background reads [18]

$$\begin{aligned} ds^2 &= e^{2\Phi}\delta_{\mu\nu}dx^\mu dx^\nu\,, \\ H &= dB = 2Q\epsilon_3\,, \\ e^{2\Phi} &= e^{2\Phi_0} + \frac{Q}{x^2}\,. \end{aligned} \tag{5.1}$$

$\mu, \nu = 0, \ldots, 3$ and $x^2 = x_\mu x^\mu$. ϵ_3 is the volume form of a unit $\mathbf{S}^3$.

We look for a potential ϕ that reproduces this background with $\mu_p = 0$. Thus, ϕ obeys the free field equation in four dimensions. Using the metric equations (4.4) we obtain

$$\partial_1 \partial_{\bar{1}} \phi = -\partial_2 \partial_{\bar{2}} \phi = e^{2\Phi_0} - 1 + \frac{Q}{x^2} \,. \tag{5.2}$$

To solve this, we assume that ϕ depends on $r_1 = |z_1|$ and $r_2 = |z_2|$. ϕ takes the form

$$\begin{aligned} \phi = {} & \frac{e^{2\Phi_0} - 1}{4}(r_1^2 - r_2^2) - \frac{Q}{4} \operatorname{Li}_2\left(-\frac{r_1^2}{r_2^2}\right) + Q \log r_1 \log r_2 - \frac{Q}{2} (\log r_2)^2 + \\ & + (k_1 + k_2 \log r_1)(k_3 + k_4 \log r_2) \,. \end{aligned} \tag{5.3}$$

Here k_1, k_2, k_3, k_4 are constant and $\operatorname{Li}_2(x)$ is the polylogarithm function satisfying

$$\frac{d}{dx} \operatorname{Li}_2(x) = -\frac{\log(1-x)}{x} \,. \tag{5.4}$$

Note that the potential depends on four free parameters. These correspond to symmetry transformations that leave the background unchanged. This is similar to Kähler transformations of ϕ in the β-string.

The solution with signature $(2, 2)$ is obtained by $r_1 \to i r_1$.

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